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Non-split geometry on products of vector bundles

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Abstract. It has recently been established that a product bundle, composed of two gauge structures, under some circumstances, possesses a geometry which does not split. Here we provide an educated extension of the above idea to products of *many* vector bundles with a distinct group structure associated with each factor fibrespace in the splice. The model employs connection 1-forms with values in a space product of Lie algebras, and therefore interlaces the various gauge structures in a non-trivial manner. Special attention is given to the structure of the geometric ghost sectors and the super-algebra they possess.

1. Introduction

A product of vector bundles, within the classical framework of gauge theories, is often contemplated as the bundle of product space fibres or, otherwise, as the bundle of fibre direct sums. Then, the splitting of the fibres causes a splitting of the corresponding geometries by trivial means: when the geometrical aspects of a single group structure are considered, those components of a geometric object that correspond to other co-existing group structures, all remain non-active. This observation is after all a simple consequence of the Leibnitz rule. For example: the absolute differential of a tensor product of two fibre bases (vector fields) splits into a sum of tensor products, each contains an absolute differential of a respective single-basis, which is, in turn, used to define the corresponding factor structure connection,

$$\begin{aligned} \mathrm{d}(e_1 \otimes e_2) &= (\mathrm{d}e_1) \otimes e_2 + e_1 \otimes \mathrm{d}(e_2) \\ &=: \omega_1(e_1) \otimes e_2 + e_1 \otimes \omega_2(e_2). \end{aligned} \tag{1}$$

In other words, the geometry of the entire splice is determined by considering horizontal transports of factor-fibre bases, one at a time.

The splitting forced upon us by the Leibnitz rule, however, is not entirely compatible with the idea of fused structures: is it possible to form a better glue of fibres, one that really fuses the geometries of the composite bundle? This is indeed possible, under certain circumstances, by exploiting a different type of connection inducement: consider again the case of a glue of two structures, with the following formal redefinition of connections:

$$\begin{aligned} \mathrm{d}(e_1 \otimes e_2) &= (\mathrm{d}e_1) \otimes e_2 + e_1 \otimes \mathrm{d}(e_2) \\ &=: \omega_1(e_1 \otimes e_2) + \omega_2(e_1 \otimes e_2). \end{aligned} \tag{2}$$

The conditions for which definition (2) can be really accepted, and which make it also meaningful, will later be elaborated. For the moment, we shall only mention that it gives rise to a non-split geometrical structure, even though the bundle itself inherently splits.

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This work contains a generalization and an elaboration of a previously proposed model [1]. The text includes two main sections, additional closing remarks, and an appendix:

• *Section 2. Pedagogical presentation of the model*. Following a concise description of the mathematical set-up (product bundle), and a listing of some useful conventions, a non-split geometry is constructed on the basis of a collection of connection 1-forms, each taking values in a space product of Lie algebras, instead of in the direct sum. These are integrated to form a single curvature of the multistructure splice. We discuss the requirements for a consistent construction, thus defining a new notion, that of algebras sealed in a representation. A somewhat different view, based on applying exterior action on product frames, is also elaborated and shown to lead to the same results. Extended covariant exterior derivatives are finally presented, and used in reconstructing some basic structural identities.

• *Section 3. Internal structure analysis*. The second part of this work deals with the geometrical properties of the multi-gauge ghost sector. The respective BRST variation laws are derived by pure geometrical means. Various aspects of the gauge-ghost extended frame are studied, including base–fibre interplay via Yang–Mills-type torsion, off-diagonal extension of Maurer–Cartan equations for product structures, and a representation of the B-fields framework as a completely equivalent description of the ghost sector. The underlying ideas and the reasonings behind are elaborated throughout. A particular duality symmetry, which is manifested at the entire gauge sector, and which enhances a super BRST structure, is finally realized.

• *Closing*. Additional related remarks are summarized in section 4. A straightforward derivation of curvature coefficients, and a few words about the consequential group structure, are finally appended.

2. The formalism of non-split splices

In the following we shall be interested in products of vector bundles. The underlying manifold M is taken to be smooth and oriented, and each factor fibrespace $(V_\alpha)_x$ is a representation space of an arbitrary-rank G_α-tensor, $G_\alpha \equiv G_\alpha(x)$, $x \in M$, is a Lie group labelled by α. Our objects of interest are geometrical forms, as well as matrix-valued forms, belonging to

$$E = \bigcup_{x\in M} \left(\bigoplus_{p=0}^{n} \bigwedge{}^{p} T_x^\star M \right) \otimes \left(\bigotimes_{\alpha=1}^{m} (V_\alpha)_x \right) =: \bigcup_{x\in M} \mathcal{B}_x \otimes \mathcal{F}_x \tag{3}$$

where $\mathcal{B}_x$ stands for a local Grassmann space of the base, and $\mathcal{F}_x$ is a space product of single-group, product-space fibres. We shall often use the term *foil*† for $\mathcal{F}_x$ and *foliar complex* to denote that particular piece of E, later to be discussed, whose geometry is claimed to be of a non-split nature.

Notation and conventions. Above, and in what follows, α and γ are labels of members of a given collection of m fibrespaces, groups, or Lie algebras; $n = \dim M$, $n_\alpha = \dim G_\alpha$, $N_\alpha = \dim V_\alpha$ for any $\alpha = 1 \ldots m$. In addition, $a_\alpha, b_\alpha, \ldots = 1 \ldots n_\alpha$ are G_α-indices, $A_\alpha, B_\alpha, \ldots = 1 \ldots N_\alpha$ are V_α-fibrespace indices, and $\mu, \nu, \ldots = 1 \ldots n$ are basespace holonomic indices. Notice that group space and fibrespace indices come with labels. Concerning bracket notation, we put

$$[\psi, \phi]_\mp = \psi\phi \mp \phi\psi \qquad \text{and} \qquad [\psi, \phi] = \psi\phi - (-1)^{\deg(\psi)\deg(\phi)}\phi\psi$$

† We use the term *foil* here instead of the term *leaf* which was used in [1] in order to avoid possible confusion with foliation theory nomenclature.

whatever ψ and ϕ, and whatever the type of product among them. Finally, the summation convention will be frequently adopted.

E then refers to a product bundle consisting of m distinct independent co-existing structures. The generating algebras $\{\mathrm{Lie}\, G_\alpha\}$ are *assumed* to carry a faithful representation ρ_α in V_α, and also to extend to the full enveloping associative algebras (so the anti-commutator representation is defined). We shall restrict ourselves to deal only with G_α-structures whose represented generators $\rho_\alpha\{L^\alpha_{a_\alpha}\}$ satisfy

$$[\rho_\alpha(L^\alpha_{a_\alpha}), \rho_\alpha(L^\alpha_{b_\alpha})]_+ \in \mathrm{span}\{\rho_\alpha(L^\alpha_{c_\alpha})\}. \tag{4}$$

The realizations are closed with respect to anti-commutation. An algebra whose represented elements close with respect to anti-commutation is said to be *sealed* in that representation. We stress that, although it is not always easy to achieve, the requirement that the algebra be sealed in a representation is obligatory for our purposes. A simple example of such an algebra is the one which generates invertible linear transformations in a vector space. For unitary structures, however, the inclusion (4) is satisfied only if the algebra is extended to include the centres.

Next we introduce a set of m G_α-induced connection 1-forms $\{\omega_\alpha\}$ all of which carry a representation ρ_E, sealed in $\mathcal{F}_x \in E$ for all $x \in M$:

$$\omega_\alpha(G_\alpha) = \rho_E(\bar{\omega}_\alpha(G_\alpha)) := \bar{\omega}^{a_1\ldots a_m}_\mu(G_\alpha)\boldsymbol{e}^\mu \rho_{a_1\ldots a_m} \tag{5}$$

$\{\bar{\omega}^{a_1\cdots a_m}(G_\alpha)\}$ are G_α-induced coefficients; the short-hand notation $\rho_{a_1\cdots a_m}$ stands for a product space of representations,

$$(\rho_{a_1\cdots a_m})^{B_1\ldots B_m}_{A_1\ldots A_m} := \bigotimes_{\gamma=1}^{m}(\rho_\gamma(L^\gamma_{a_\gamma}))^{B_\gamma}_{A_\gamma} \tag{6}$$

and the set of basespace monomials $\{\boldsymbol{e}^\mu\} \in T^\star M$ span a local basis for the cotangent bundle of 1-forms. Note that, in general, $\bar{\omega}^{a_1\ldots a_m}(G_\alpha) \neq \bar{\omega}^{a_1\ldots a_m}(G_\gamma)$ for $\alpha \neq \gamma$, leading to $\omega_\alpha \neq \omega_\gamma$. By construction (see (16) for details), each element of the collection $\{\omega_\alpha\}$ obeys the following laws of gauge:

$$\forall g_\alpha \in G_\alpha : \begin{cases} \omega_\gamma \mapsto g_\alpha(\omega_\gamma + \mathrm{d})g_\alpha^{-1} & \alpha = \gamma \\ \omega_\gamma \mapsto g_\alpha \omega_\gamma g_\alpha^{-1} & \alpha \neq \gamma \end{cases} \tag{7}$$

where d stands for exterior differentiation on M, and the actions of the g's are given by means of matrix multiplication. Each ω_α, therefore, transforms as a connection with respect to its inducing group G_α, while behaving as a tensor with respect to the rest of the groups in the collection.

We shall now state our *fundamental assertion*:

There exists a complex in E whose geometry does not split even though E itself, being a product bundle, inherently splits. We call it the *foliar complex* (FC) associated with the product bundle E.

The set of connection 1-forms introduced above solely determines the structure of FC. This is best seen by considering the curvature 2-form which we propose to associate with the foliar complex,

$$\mathcal{R}_{\mathrm{FC}}(\{\omega\}) = \sum_{\alpha,\gamma=1}^{m}(\mathrm{d}\omega_\alpha + \omega_\alpha \wedge \omega_\gamma). \tag{8}$$

To see that this is indeed a proper curvature, one follows two steps: first, one verifies that the algebraic structure is preserved by the construction, namely, that $\mathcal{R}_{\mathrm{FC}}$ also takes values $\in \otimes_\gamma$ (Lie G_γ). This, however, follows directly from the fact that $\rho(\text{Lie}\, G)$ for any $G \in \{G\}$ closes with respect to anti-commutation,

$$\rho(L)\rho(L) = \tfrac{1}{2}[\rho(L), \rho(L)]_+ + \tfrac{1}{2}[\rho(L), \rho(L)]_- \subset \text{span}\,\{\rho(\text{Lie}\, G)\}. \tag{9}$$

Consequently, no matter how many products of generators are found in each product term $\omega_\alpha \wedge \omega_\gamma$, assignment (9) guarantees that the resulting algebraic expansion will always lay in $\otimes_\gamma$ (Lie G_γ). And since $\mathrm{d}\rho_E(\cdot) = \rho_E(\mathrm{d}\cdot)$, we finally conclude

$$\begin{aligned}\mathcal{R}_{\text{FC}} = \mathcal{R}_{\mu\nu}(\rho_E(\bar{\omega}))\boldsymbol{e}^\mu \wedge \boldsymbol{e}^\nu &= \sum_{\alpha,\gamma=1}^{m} (\mathrm{d}\rho_E(\bar{\omega}_\alpha) + \rho_E(\bar{\omega}_\alpha) \wedge \rho_E(\bar{\omega}_\gamma)) \\ &= \sum_{\alpha,\gamma=1}^{m} (\rho_E(\mathrm{d}\bar{\omega}_\alpha) + \rho_E(\bar{\omega}_\alpha \wedge \bar{\omega}_\gamma)) = \rho_E(\mathcal{R}_{\mu\nu}(\bar{\omega}))\boldsymbol{e}^\mu \wedge \boldsymbol{e}^\nu. \end{aligned} \tag{10}$$

A detailed derivation of the coefficients $\{f^{(1\ldots m)_-}_{a_1\ldots a_m\, b_1\ldots b_m\, c_1\ldots c_m}\}$ involved in the explicit expression for $\rho_E(\bar{\omega}_\alpha \wedge \bar{\omega}_\gamma)$, and a comment on the resulting group structure of FC, are given in the appendix.

Second, one shows that $\mathcal{R}_{\text{FC}}$ is multilinear. Under the action of some representative group, say G_α, $\mathcal{R}_{\text{FC}}$ decomposes into $2m - 1$ terms,

$$\mathrm{d}\omega_\alpha + \omega_\alpha \wedge \omega_\alpha \tag{11}$$

$$+\sum_{\gamma\neq\alpha}(\mathrm{d}\omega_\gamma + \omega_\alpha \wedge \omega_\gamma + \omega_\gamma \wedge \omega_\alpha) \tag{12}$$

$$+\sum_{\gamma\neq\alpha}\omega_\gamma \wedge \omega_\gamma \tag{13}$$

each of which transform linearly and in an independent manner with respect to that particular G_α. This, however, holds for any $G_\alpha \in \{G_\gamma\}$. Thus $\mathcal{R}_{\text{FC}}$ is linear with respect to all the G's. And since actions of different groups commute, $\mathcal{R}_{\text{FC}}$ is linear also with respect to a *simultaneous* action of any subcollection of groups. Thus the claim has been established.

Two comments are in order.

(1) The foliar gauge model exhibits indifference to global rescaling in the spaces of connections: $\omega_\alpha \mapsto c(\alpha)\omega_\alpha$, where the m $c(\alpha)$'s are x-independent $(\times_\gamma G_\gamma)$-scalar scaling parameters. The curvature then acquires a generalized form

$$\mathcal{R}_{\text{FC}}(\{\omega\}, \{c\}) = \sum_{\alpha,\gamma=1}^{m} (c(\alpha)\,\mathrm{d}\omega_\alpha + c(\alpha\gamma)\omega_\alpha \wedge \omega_\gamma) \tag{14}$$

with $c(\alpha\gamma) = c(\alpha)c(\gamma) = c(\gamma\alpha)$. Note that decomposition (11)–(13) of $\mathcal{R}_{\text{FC}}$ into G_α-linear pieces still remains valid after the connections are rescaled. The form of $\mathcal{R}_{\text{FC}}$ as was given by (8) should, therefore, be replaced by the more general one, (14). In this case, given a set of connection 1-forms $\{\omega_\alpha\}$, the phase space of non-zero couplings spanned by $\{c(\alpha)\}$ specifies a class of continuously tuned curvatures. In other words, each curvature $\in$ FC is given up to m continuous parameters with respect to which it can be adjusted.

(2) We notice that $(\sum_\alpha \omega_\alpha)$ can be regarded as a single connection, having the property of supporting simultaneously many gauges:

$$\forall\,\gamma \text{ and } \forall g_\gamma \in G_\gamma : \left(\sum_\alpha \omega_\alpha\right) \mapsto g_\gamma\left(\left(\sum_\alpha \omega_\alpha\right) + \mathrm{d}\right)g_\gamma^{-1}. \tag{15}$$

Therefore, $(\sum_\alpha \omega_\alpha)$ underlies a generic formation of gauge, for which m distinct coexisting structures are intertwined, and whose associated multilinear curvature acquires a 'single-structure' form, $\mathcal{R}_{\text{FC}} = \mathrm{d}(\sum_\alpha \omega_\alpha) + (\sum_\alpha \omega_\alpha) \wedge (\sum_\alpha \omega_\alpha)$. This interpretation also complies with other aspects of the theory later to be discussed.

As a clarifying illustration consider briefly the case of two gauge groups, G_1 and G_2, and put $\omega_1 = \omega$ and $\omega_2 = \varphi$. The two-folium curvature $\mathcal{R}_{\mathrm{FC}} = \mathrm{d}(\omega+\varphi)+(\omega+\varphi)\wedge(\omega+\varphi)$ supports two kinds of manifestly covariant decompositions:

(1) manifesting G_1-covariance

$$\mathcal{R}_{\mathrm{FC}} = (\mathrm{d}\omega + \omega\wedge\omega) + (\mathrm{d}\varphi + \omega\wedge\varphi + \varphi\wedge\omega) + (\varphi\wedge\varphi)$$

(2) manifesting G_2-covariance

$$\mathcal{R}_{\mathrm{FC}} = (\mathrm{d}\varphi + \varphi\wedge\varphi) + (\mathrm{d}\omega + \varphi\wedge\omega + \omega\wedge\varphi) + (\omega\wedge\omega).$$

In conclusion, $\mathcal{R}_{\mathrm{FC}}$ is a two-group linear object. A G_1-decomposition is associated with a *reduced* G_1-bundle, where only G_1 is activated and G_2 is frozen. Over the reduced G_1-bundle, ω represents a connection, whereas φ is a coframe element with values in Lie$\,(G_1)$. There, $R_\omega = \mathrm{d}\omega + \omega\wedge\omega$ is the reduced curvature whose coefficients are given by $\{f^{(12)-}_{a_1a_2b_1b_2c_1c_2}\}$ and $T_\omega(\varphi) = \mathrm{d}\varphi + [\omega,\varphi] \equiv D_\omega\varphi$ is the counterpart torsion. Reversing the roles played by ω and φ, a G_2-covariant decomposition associates with a reduced G_2-bundle, where this time G_2 is activated and G_1 is frozen, consisting of a G_2-curvature $R_\varphi = \mathrm{d}\varphi+\varphi\wedge\varphi$, with the same characteristic two-folium coefficients as before, and whose counterpart torsion is given by $T_\varphi(\omega) = \mathrm{d}\omega + [\varphi,\omega] \equiv D_\varphi\omega$.

The prime motivation behind the concept of non-split geometry, as was already implied in the introduction, comes from the observation that the geometrical framework generated by taking the tensor product of single-fibre horizontal transports need not be the same as that obtained by employing a horizontal transport on tensor products of fibres. Indeed, in contrast with the former case, the latter interlaces the geometries of the individual bundles associated with each factor structure in the splice. Consider the absolute differential of a foliar frame field $\boldsymbol{e}_{A_1\ldots A_m}(x) = \otimes_{\alpha=1}^m \boldsymbol{e}^\alpha_{A_\alpha}(x)$,

$$\begin{aligned}\mathrm{d}\boldsymbol{e}_{A_1\ldots A_m} &= \mathrm{d}\bigotimes_{\alpha=1}^{m}\boldsymbol{e}^\alpha_{A_\alpha} = \sum_{\gamma=1}^{m}\left(\left(\sum_{B_\gamma=1}^{N_\gamma}\varpi_{\gamma}{}^{B_\gamma}{}_{A_\gamma}\boldsymbol{e}^\gamma_{B_\gamma}\right)\bigotimes_{\alpha\neq\gamma}\boldsymbol{e}^\alpha_{A_\alpha}\right)\\ &:= \sum_{\gamma=1}^{m}\left(\sum_{B_1\ldots B_m} -(\rho_E(\bar\omega_\gamma))^{B_1\ldots B_m}_{A_1\ldots A_m}\boldsymbol{e}_{B_1\ldots B_m}\right).\end{aligned} \tag{16}$$

A linear expansion of the γth factor-frame results in a corresponding set of coefficients ϖ_γ which, in turn, induces an associated connection 1-form $\omega_\gamma := \rho_E(\bar\omega_\gamma)$ with values $\in \otimes_\alpha$ (Lie$\,G_\alpha$). The collection $\{\omega_\gamma\}$ is naturally identified with that of definitions (5) and (6). An overall gauge transformation applied simultaneously to all factor fibrespaces $\in E$,

$$\forall\gamma \text{ and } \forall g_\gamma \in G_\gamma : \bigotimes_{\gamma=1}^{m}\boldsymbol{e}^\gamma_{A_\gamma} \mapsto \bigotimes_{\gamma=1}^{m} g_\gamma{}^{B_\gamma}{}_{A_\gamma}\boldsymbol{e}^\gamma_{B_\gamma} \tag{17}$$

uniquely dictates the gauge sector gauge laws of (7). Otherwise definition (16) would not be automatically satisfied as an identity.

Definition (16) is conveniently abbreviated as $\mathrm{d}\boldsymbol{e}_{A_1\ldots A_m} = -\sum_\alpha(\omega_\alpha\boldsymbol{e})_{A_1\ldots A_m}$. Now, additional application of d immediately gives

$$\begin{aligned}&\mathrm{dd}\boldsymbol{e}_{A_1\ldots A_m} = -(\mathcal{R}_{\mathrm{FC}}\boldsymbol{e})_{A_1\ldots A_m}\\ &(= -(\rho_E(\mathcal{R}_{\mathrm{FC}}(\bar\omega)))^{B_1\ldots B_m}_{A_1\ldots A_m}\boldsymbol{e}_{B_1\ldots B_m})\end{aligned} \tag{18}$$

which is seen to measure the overall effect generated by dragging a foil horizontally along an infinitesimal parallelogram on M. Evidently, the result differs from the sum of single-fibre closed tracks. Therefore, speaking in terms of Whitney constructions, the foliar curvature of an m-splice is different from the two types of sums one usually constructs from single-structure curvatures, $\mathcal{R}^{(1)}_\alpha = \mathrm{d}\omega^{(1)}_\alpha + \omega^{(1)}_\alpha \wedge \omega^{(1)}_\alpha$, namely:

$$\mathcal{R}^{(m)}_{\mathrm{FC}} \neq \sum_{\alpha=1}^{m} \left(\mathcal{R}^{(1)}_\alpha \bigotimes_{\gamma\neq\alpha} I_\gamma \right) \qquad \mathcal{R}^{(m)}_{\mathrm{FC}} \neq \bigoplus_{\alpha=1}^{m} \mathcal{R}^{(1)}_\alpha. \tag{19}$$

Next we introduce linear exterior derivatives suitable for sections of the foliar complex. For this purpose we make a distinction between vector-valued *folium forms* ($\equiv \Psi_{\mathrm{V}}$) and tensor-valued ones ($\equiv \Psi_{\mathrm{T}}$). The former quantities are by construction $(\times_\gamma G_\gamma)$-vectors, and the latter ones are by construction $(\times_\gamma G_\gamma)$-tensors. These are (in addition to functions) the legitimate geometric residents of FC, whose linear structure survive derivations: in precise terms, $\mathcal{D}\Psi_{\mathrm{V}} := \mathrm{d}\Psi_{\mathrm{V}} + \sum_\alpha \omega_\alpha \wedge \Psi_{\mathrm{V}}$ is a folium vector, whereas

$$\mathcal{D}\Psi_{\mathrm{T}} := \mathrm{d}\Psi_{\mathrm{T}} + \sum_\alpha (\omega_\alpha \wedge \Psi_{\mathrm{T}} + (-1)^{\deg(\Psi_{\mathrm{T}})+1} \Psi_{\mathrm{T}} \wedge \omega_\alpha) \tag{20}$$

is a folium tensor. The proofs are just standard ones. One first shows that $\mathcal{D}\Psi_{\mathrm{V}}$ and $\mathcal{D}\Psi_{\mathrm{T}}$ are G-linear with respect to any $G \in \{G\}$. Then one shows that both derivatives satisfy the graded Leibnitz rule with respect to exterior multiplication of folium forms†.

In terms of these covariant exterior derivatives, the curvature $\mathcal{R}_{\mathrm{FC}}$ can now be constructed via $[\mathcal{D}, \mathcal{D}]\Psi_{\mathrm{V}} = 2\mathcal{R}_{\mathrm{FC}} \wedge \Psi_{\mathrm{V}}$, or else, via $[\mathcal{D}, \mathcal{D}]\Psi_{\mathrm{T}} = 2[\mathcal{R}_{\mathrm{FC}}, \Psi_{\mathrm{T}}]$. Moreover, by the Jacobi identity we have

$$0 = [\mathcal{D}, [\mathcal{D}, \mathcal{D}]]\Psi_{\mathrm{T}} = 2\mathcal{D}[\mathcal{R}_{\mathrm{FC}}, \Psi_{\mathrm{T}}] - 2[\mathcal{R}_{\mathrm{FC}}, \mathcal{D}\Psi_{\mathrm{T}}]$$

and $\mathcal{D}\mathcal{R}_{\mathrm{FC}} = 0$ follows immediately. For a multi-Lie-valued form Ψ_{T} we obtain $\mathcal{D}\mathcal{T}_{\mathrm{FC}} = [\mathcal{R}_{\mathrm{FC}}, \Psi_{\mathrm{T}}]$ with $\mathcal{T}_{\mathrm{FC}} := \mathcal{D}\Psi_{\mathrm{T}}$ being the torsion on FC. We shall elaborate on the torsion matter later in the next section. Evidently, the two Bianchi identities, $\mathcal{D}\mathcal{T}_{\mathrm{FC}} = [\mathcal{R}_{\mathrm{FC}}, \Psi_{\mathrm{T}}]$ and $\mathcal{D}\mathcal{R}_{\mathrm{FC}} = 0$, automatically imply $\mathcal{D}\mathcal{D}\mathcal{T}_{\mathrm{FC}} = [\mathcal{R}_{\mathrm{FC}}, \mathcal{T}_{\mathrm{FC}}]$. In general, the action of a pth power of $\mathcal{D}\mathcal{D}$ on $\Phi_{\mathrm{T}} := \mathcal{D}\Psi_{\mathrm{T}}$ produces a p-nested even commutator of the type $[\mathcal{R}_{\mathrm{FC}}, [\ldots, [\mathcal{R}_{\mathrm{FC}}, \Phi_{\mathrm{T}}] \ldots]]$. Now, a polynomial in $\mathcal{D}\mathcal{D}$ applied to Φ_{T} obviously terminates at the power of $[n/2] - \deg \Phi_{\mathrm{T}}$. However, at the limit where $n \to \infty$, but keeping $\dim(\otimes_\alpha V_\alpha)$ finite, the infinite sum of nested commutators can be converted into enveloping exponentials (by means of induced representations) and one can formally write

$$\exp(\mathcal{D}\mathcal{D})\Phi_{\mathrm{T}} = (\exp(\mathcal{R}_{\mathrm{FC}}))\Phi_{\mathrm{T}}(\exp(-\mathcal{R}_{\mathrm{FC}})).$$

3. Ghosts, torsion, and the entire FC BRST super structure

Our next aim is to explore the geometry induced along the 'internal' directions. For this propose, consider the following set of mutually-independent *horizontal* translations $\omega_\alpha \to \omega_\alpha + \Omega_\alpha$, where the shifts $\{\Omega_\alpha\}$ are linear with respect to all the G's. Consequently $\omega_\alpha + \Omega_\alpha$ transforms according to (19) but it is constructed such that it can never be gauge connected back to ω_α; thus, each translation displays a bijection between two gauge-inequivalent orbits in moduli space. In general, the shifted connections correspond to a different curvature. This, however, can be avoided by extending the original basespace such that it also includes the angles associated with all the gauge groups, from now on considered as additional independent variables [2]. Namely, enlarging the basespace is expected to compensate for making the shifts.

Each set of angles $\{\phi^{a_\alpha}(x)\}$, coordinating $G_\alpha(x)$, is naturally supplied with a coboundary-type operator δ_α, in complete analogy with the exterior derivative d on M. More

† In contrast with an incorrect statement that was given in [1], covariance will not be supported only in cases where factor-terms in a product are not pure folium forms.

specifically, each group $G_\alpha(x)$, at any $x \in M$, is associated with a Grassmann space, graded by δ_α, over which Ω_α is taken as a 1-form. This can be established as follows. In terms of local coordinates, one puts $\mathrm{d} := \mathrm{d}x^\mu(\partial/\partial x^\mu)$ and $\delta_\alpha := \mathrm{d}\phi^{a_\alpha}(\partial/\partial\phi^{a_\alpha})$. By construction, all exterior derivatives anti-commute: $\mathrm{d}\delta_\alpha + \delta_\alpha\,\mathrm{d} = \delta_\alpha\delta_\gamma + \delta_\gamma\delta_\alpha = 0$. Evidently,

$$\delta_\alpha\phi^{a_\alpha} = \mathrm{d}\phi^{b_\alpha}(\partial\phi^{a_\alpha}/\partial\phi^{b_\alpha}) = \mathrm{d}\phi^{a_\alpha} = (\partial\phi^{a_\alpha}/\partial x^\mu)\,\mathrm{d}x^\mu$$

and, moreover, since for each α the ϕ's smoothly depend on x, and since the inverse dependence is also assumed, we also have $\mathrm{d}\phi^{a_\alpha} = (\partial\phi^{a_\alpha}/\partial\phi^{a_\gamma})\,\mathrm{d}\phi^{a_\gamma}$. Thus, any differential 1-form $\Omega_{a_\alpha}\,\mathrm{d}\phi^{a_\alpha}$ which takes its values in $\otimes_\gamma$ (Lie G_γ) induces a linear shift (not necessarily horizontal) via a differential transition,

$$\rho_E(\Omega_{a_\alpha})\,\mathrm{d}\phi^{a_\alpha} = \rho_E(\Omega_{a_\alpha})\frac{\partial\phi^{a_\alpha}}{\partial x^\mu}\,\mathrm{d}x^\mu =: \rho_E((\Omega_\alpha)_\mu)\,\mathrm{d}x^\mu =: \Omega_\alpha. \tag{21}$$

Owing to their algebraic properties, which we shall soon fully reveal, we identify $\{\delta_\alpha\}$ with FC BRST operators, and the *horizontal* shifts $\{\Omega_\alpha\}$ with folium '*ghosts*' [2]. Following these identifications, a term of the form $(\partial\phi^{a_\alpha}/\partial x^{\mu_2})(\partial\phi^{b_\alpha}/\partial x^{\mu_2})$, which appears in any basespace implementation of indices of a horizontal geometric object, corresponds to a ghost number N_α associated with that object.

Letting all of our bundle objects, in particular the connections and the shifts, depend on all group angles, thus extending the basespace such that it includes the group manifolds as well, requires a proper modification of the covariant exterior derivative, $\mathcal{D} \to \hat{\mathcal{D}}$:

$$\begin{aligned}\hat{\mathcal{D}}\Psi_\mathrm{T} := \mathrm{d}\Psi_\mathrm{T} + \sum_{\alpha=1}^{m}(&\delta_\alpha\Psi_\mathrm{T} + \omega_\alpha\wedge\Psi_\mathrm{T} + (-1)^{\deg(\Psi_\mathrm{T})+1}\Psi_\mathrm{T}\wedge\omega_\alpha \\ &+\Omega_\alpha\wedge\Psi_\mathrm{T} + (-1)^{\deg(\Psi_\mathrm{T})+1}\Psi_\mathrm{T}\wedge\Omega_\alpha).\end{aligned} \tag{22}$$

In particular, two successive applications of $\hat{\mathcal{D}}$ on a generic Ψ_T yields

$$\begin{aligned}\hat{\mathcal{D}}\hat{\mathcal{D}}\Psi_\mathrm{T} = [\mathcal{R},\Psi] &+ \left[\sum_\alpha \mathcal{D}\Omega_\alpha, \Psi_\mathrm{T}\right] + \left[\sum_{\alpha,\gamma}\delta_\gamma\omega_\alpha, \Psi_\mathrm{T}\right] \\ &+\left[\sum_{\alpha,\gamma}\delta_\gamma\Omega_\alpha, \Psi_\mathrm{T}\right] + \left[\sum_{\alpha,\gamma}\Omega_\alpha\wedge\Omega_\gamma, \Psi_\mathrm{T}\right].\end{aligned} \tag{23}$$

The imposed shifts, followed by enlarging the basespace, are required to cause no geometrical impact: the curvature remains the same, thus the extra four terms in (23) must sum up to zero. Comparing terms of equal Grassmann grade we find $m(m+1)$ BRST variation laws,

$$\delta_{[\alpha}\Omega_{\gamma]_+} = -\Omega_{[\alpha}\wedge\Omega_{\gamma]_+} \tag{24}$$

$$\delta_\alpha\left(\sum_{\gamma=1}^{m}\omega_\gamma\right) = -\mathcal{D}\Omega_\alpha. \tag{25}$$

Equation (24) implies that $\delta_\alpha\Omega_\gamma$ equals either $-\Omega_\gamma\wedge\Omega_\alpha$ or $-\Omega_\alpha\wedge\Omega_\gamma$. Without loss of generality we take the former possibility. It is easily verified that any of the operators $\{\delta_\alpha\}$ square to zero on both Ω_γ and $\sum_\gamma\omega_\gamma$. Note also that the sum as a whole, $\sum_\gamma\omega_\gamma =: \omega_\mathrm{FC}$, and not each particular summand, possesses a definite transformation law; each δ-variation detects a *single-gauge* connection in a theory in which the curvature can be cast in a *single-structure* form, $\mathcal{R}_\mathrm{FC} = \mathrm{d}\omega_\mathrm{FC} + \omega_\mathrm{FC}\wedge\omega_\mathrm{FC}$.

Ghosts are seen also from a different point of view. The same methods that have been used in reproducing sets of connection coefficients are also suitable for ghost coefficients. Following definition (16), folium ghosts of type γ are generated by performing the variation

$$\delta_\gamma e_{A_1\ldots A_m} = \left(\sum_{B_\gamma=1}^{N_\gamma} \epsilon^{B_\gamma}_{\gamma A_\gamma} e^{\gamma}_{B_\gamma}\right) \bigotimes_{\alpha\neq\gamma} e^{\alpha}_{A_\alpha} =: -(\Omega_\gamma)^{B_1\ldots B_m}_{A_1\ldots A_m} e_{B_1\ldots B_m} \tag{26}$$

where we used $e_{A_1\ldots A_m} = \otimes_{\alpha=1}^{m} e^{\alpha}_{A_\alpha}(x,\phi_\alpha)$, and therefore $\delta_\gamma e^{\alpha}_{A_\alpha}(x,\phi_\alpha) = 0$ for $\alpha \neq \gamma$. Definition (26) will be conveniently abbreviated as $\delta_\gamma e = -\Omega_\gamma e$. Let us write for a combined gauge transformation $g_E(\{\phi(x)\}) = \prod_\alpha g_\alpha(\phi_\alpha(x))$. Then, a proposition of the form $\Omega_\gamma = g_E^{-1}\delta_\gamma g_E$ [2] *cannot* be compatible, nor consistent, with our adopted guidelines: it fails to be horizontal because it is the internal part of an (extended) pure gauge and, furthermore, from $\delta_\gamma e' = -\Omega'_\gamma e'$,

$$\delta_\gamma(g_E e) = -g_E\Omega_\gamma g_E^{-1} g_E e = -g_E\Omega_\gamma e = g_E\delta_\gamma e \Rightarrow \delta_\gamma g_E = 0. \tag{27}$$

Instead, $\{\Omega_\gamma\}$, being the difference between two gauge-inequivalent orbits, should be determined by the structure of the moduli space of FC-connection 1-forms†. Next, let us rederive equations (24) and (25):

$$0 = (\delta_\gamma\, \mathrm{d} + \mathrm{d}\delta_\gamma)e = -\delta_\gamma\left(\sum_\alpha \omega_\alpha e\right) - \mathrm{d}(\Omega_\gamma e)$$

$$= \left(-\delta_\gamma \sum_\alpha \omega_\alpha - \mathcal{D}\Omega_\gamma\right)e \Rightarrow \delta_\gamma\omega_{\mathrm{FC}} = -\mathcal{D}\Omega_\gamma \tag{28}$$

$$0 = (\delta_\alpha\delta_\gamma + \delta_\gamma\delta_\alpha)e = -(\delta_{[\alpha}\Omega_{\gamma]_+} + \Omega_{[\gamma}\wedge\Omega_{\alpha]_+})e \Rightarrow \delta_{[\alpha}\Omega_{\gamma]_+} = -\Omega_{[\gamma}\wedge\Omega_{\alpha]_+}. \tag{29}$$

Equations (29) ($\equiv$(24)) are an extension to the Maurer–Cartan equations for ghosts on product bundles; the 'off-diagonal' ones are cross-fibre interferences. As in the case of a single gauge, extended Maurer–Cartan equations reflect the absence of curvature on the product-group manifold.

The quantity Ω_γ is a coframe element of E with values in $\otimes_\alpha$ (Lie G_α). The bi-graded object $\mathcal{D}\Omega_\gamma = -\delta_\gamma\omega_{\mathrm{FC}} =: \mathcal{T}_\gamma$ is consequently a tensor-valued *torsion* 2-form element $\in$ FC. Inherited from $[\delta_\alpha, \delta_\gamma]_+ = 0$, it satisfies $\delta_\alpha\mathcal{T}_\gamma + \delta_\gamma\mathcal{T}_\alpha = 0$, and the Bianchi identity $\mathcal{D}\mathcal{T}_\gamma = [\mathcal{R}_{\mathrm{FC}}, \Omega_\gamma]$ holds. Moreover, because $\mathcal{D}$ is linear, $\sum_\gamma \mathcal{T}_\gamma =: \mathcal{T}_{\mathrm{FC}}$ accumulates the overall torsion. Setting $\sum_\gamma \Omega_\gamma =: \Omega_{\mathrm{FC}}$, we finally obtain (see end of section 2):

$$\mathcal{D}\mathcal{T}_{\mathrm{FC}} = [\mathcal{R}_{\mathrm{FC}}, \Omega_{\mathrm{FC}}] \overset{(\mathcal{D}\mathcal{R}_{\mathrm{FC}}=0)}{\Longrightarrow} \mathcal{D}\mathcal{D}\mathcal{T}_{\mathrm{FC}} = [\mathcal{R}_{\mathrm{FC}}, \mathcal{T}_{\mathrm{FC}}]. \tag{30}$$

Apparently, $\mathcal{T}_{\mathrm{FC}}$ is larger than the sum of all single-structure torsion 2-forms, because

$$\begin{aligned}\mathcal{T}_{\mathrm{FC}} &= -\sum_{\gamma,\alpha}\delta_\gamma\omega_\alpha \\ &= -\sum_\gamma \delta_\gamma\omega_\gamma - \sum_{\alpha\neq\gamma}\delta_\gamma\omega_\alpha \\ &= \sum_\gamma \mathcal{T}^{(1)}_\gamma + \text{cross-fibre contributions}.\end{aligned} \tag{31}$$

A prior knowledge of the extended Maurer–Cartan equations pins down an equivalent (but not self-contained) description for the ghost sector. Let us define $m \times m$ bosonic quantities, $B_{\alpha\gamma} = \delta_\alpha\Omega_\gamma = -\Omega_\gamma\wedge\Omega_\alpha$. Obviously we have $\delta_\alpha B_{\alpha\gamma} = 0$ because $B_{\alpha\gamma}$ is by construction δ_α-exact. On the other hand, we have $\delta_\gamma B_{\alpha\gamma} = -[\Omega_\gamma, B_{\alpha\gamma}]$, from which

† Thus, our ghosts are not very much 'ghost-like'; they really possess physical content.

$\delta_\gamma \delta_\gamma B_{\alpha\gamma} = 0$ follows immediately. According to these variation laws we are dealing here with B-fields: the m diagonal ones $B_{\alpha\alpha}$ represent the sector of the decoupled factor-structure ingredients; the remaining $m(m-1)$ off-diagonal ones are fibre intertwining effects. The B-field description, however, is not entirely self-contained since the corresponding BRST variations are 'doped' with the ghosts themselves. On the other hand, we have the situation that the B-fields are now by no means auxiliary degrees of freedom. Rather, they are composites made of ghost–ghost pairs.

The variation laws (24) and (25) are manifestly invariant under a duality transformation which is realized by pair-permutation of labels $\alpha \leftrightarrow \gamma$ applied simultaneously to both equations. As for the B-fields, duality manifests itself via transposition. This provides one with an *arbitrary* classification into ghost–antighost pairs, and the corresponding pairs of BRST and anti-BRST operators. Consider for example the two-folium case $(\alpha, \gamma = 1, 2)$ and put $\delta_1 = \delta$, $\delta_2 = \bar{\delta}$, $\Omega_1 = \Omega$, $\Omega_2 = \Phi$, $\omega_1 = \omega$, and $\omega_2 = \varphi$. From (24) and (25) we have

$$\begin{aligned} &\delta\Omega = -\Omega \wedge \Omega \qquad \delta\Phi = -\Phi \wedge \Omega \qquad \delta(\omega + \varphi) = -\mathcal{D}\Omega \\ &\bar{\delta}\Phi = -\Phi \wedge \Phi \qquad \bar{\delta}\Omega = -\Omega \wedge \Phi \qquad \bar{\delta}(\omega + \varphi) = -\mathcal{D}\Phi. \end{aligned} \tag{32}$$

In particular, $\delta\Phi + \bar{\delta}\Omega = -[\Phi, \Omega]$. A simultaneous exchange $\delta \leftrightarrow \bar{\delta}$ and $\Omega \leftrightarrow \Phi$ transforms the upper triad of (32) into the lower one, and *vice versa*; thus a duality symmetry is realized. Otherwise, we set $B = -\Phi \wedge \Omega$ and $\bar{B} = -\Omega \wedge \Phi$, whose variation properties are easily read-off from (32),

$$\begin{aligned} &\delta B = 0 \qquad \delta\bar{B} = -[\Omega, \bar{B}] \\ &\bar{\delta}\bar{B} = 0 \qquad \bar{\delta}B = -[\Phi, B] \end{aligned} \tag{33}$$

(whereby $\delta\bar{\delta}B$ and $\bar{\delta}\delta\bar{B}$ vanish independently). Duality now maps a B-field into its dual one, $\bar{B}$, and the upper pair of (33) is mapped into the lower one.

4. Additional remarks

An obligatory requirement of our model for gluing gauge structures is, of course, condition (4) which puts severe limitations at the level of the algebra and on the representation spaces which we use. However, in cases where (4) is not strictly fulfilled, we can still look for appropriate extensions of the algebra such that (4) will be formally satisfied. Central extensions involved in the glue of two unitary structures were shown to generate sectors of a non-split geometrical nature as well as the decoupled (modular part of the) factor structures. In particular, the latter are seen to be totally autonomous. It is exactly for this reason that one may deal with single structures of this type, without caring much for what really happens in their geometrical periphery. This means that a physical theory which is based on the geometrical framework of many (*a priori* distinct) coexisting SU(N) structures, will not be directly affected by the existence of peripheral non-split complexes.

Consider for example an SU(2) × SU(3) composition: one should first convert to centrally-extended algebras in order to establish a suitable foliar framework, namely, to work with a U(2) × U(3) splice instead [1]. This was seen to generate a distinguished geometrical sector built of an autonomous SU(2) structure, coexistent with an autonomous SU(3) structure; pure SU(2) gluons carry no colour charge, pure SU(3) gluons carry no weak charge. There still exists, however, a non-split piece, whose gauge gluons carry colour and weak charges simultaneously, such as the leptoquarks of an SU(5) grand unified theory. Of course, in contrast with the SU(5) case, the resulting modular factor structures have nothing to do with the breakdown of a grand-group symmetry. They just split-off, leaving behind a residual leptoquark sector. However, these issues are not within the scope of the present work.

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Appendix. Computing $\mathcal{R}_{\mathbf{FC}}$ coefficients

By assumption, for each label γ we have

$$[\rho_\gamma(L^\gamma_{a_\gamma}), \rho_\gamma(L^\gamma_{b_\gamma})]_\mp = \sum_{c_\gamma=1}^{n_\gamma} f^{(\gamma)\mp}_{a_\gamma b_\gamma c_\gamma} \rho_\gamma(L^\gamma_{c_\gamma}) \tag{34}$$

where $\{f^{(\gamma)-}_{a_\gamma b_\gamma c_\gamma}\}$ are the structure constants of the group G_γ, and $\{f^{(\gamma)+}_{a_\gamma b_\gamma c_\gamma}\}$ are the Clebsch–Gordon coefficients of the $V_\gamma \times V_\gamma \mapsto V_\gamma$ homomorphisms. Consider for the moment the product $\omega \wedge \omega$ for any $\omega \in \{\omega\}$:

$$\begin{aligned}\omega \wedge \omega &= \sum_{\{a\}}^{\{n\}} \sum_{\{b\}}^{\{n\}} \bar{\omega}^{\{a\}}_{[\mu} \rho_{\{a\}} \bar{\omega}^{\{b\}}_{\nu]_-} \rho_{\{b\}} e^\mu \wedge e^\nu \\ &= \frac{1}{2} \sum_{\{a\}}^{\{n\}} \sum_{\{b\}}^{\{n\}} \bar{\omega}_{\{a\}} \wedge \bar{\omega}_{\{b\}} [\rho_{\{a\}}, \rho_{\{b\}}]_- \end{aligned} \tag{35}$$

where $\{a\}$ stands for the sequence $a_1 \ldots a_m$, etc. Now, according to (34), for each factor algebra $\in \otimes_\gamma$ (Lie G_γ) we have,

$$\begin{aligned}\rho_\gamma(L^\gamma_{a_\gamma})\rho_\gamma(L^\gamma_{b_\gamma}) &= \frac{1}{2}[\rho_\gamma(L^\gamma_{a_\gamma}), \rho_\gamma(L^\gamma_{b_\gamma})]_+ + \frac{1}{2}[\rho_\gamma(L^\gamma_{a_\gamma}), \rho_\gamma(L^\gamma_{b_\gamma})]_- \\ &= \frac{1}{2} \sum_{c_\gamma=1}^{n_\gamma} (f^{(\gamma)+}_{a_\gamma b_\gamma c_\gamma} + f^{(\gamma)-}_{a_\gamma b_\gamma c_\gamma}) \rho_\gamma(L^\gamma_{c_\gamma}).\end{aligned} \tag{36}$$

Moreover, since $[\rho_{\{a\}}, \rho_{\{b\}}]_\mp (\equiv [\rho_{a_1\ldots a_m}, \rho_{b_1\ldots b_m}]_\mp)$ can be rewritten in terms of tensor products of representations of ordered pairs,

$$[\rho_{\{a\}}, \rho_{\{b\}}]_\mp = \bigotimes_{\gamma=1}^{m} \rho_\gamma(L^\gamma_{a_\gamma} L^\gamma_{b_\gamma}) \mp \bigotimes_{\gamma=1}^{m} \rho_\gamma(L^\gamma_{b_\gamma} L^\gamma_{a_\gamma}) \tag{37}$$

we have

$$[\rho_{\{a\}}, \rho_{\{b\}}]_\mp = \sum_{\{c\}}^{\{n\}} f^{(1\ldots m)\mp}_{\{a\}\{b\}\{c\}} \rho_{\{c\}} \tag{38}$$

with the coefficients $\{f^{(1\ldots m)\mp}_{\{a\}\{b\}\{c\}} \equiv f^{(1\ldots m)\mp}_{a_1\ldots a_m\, b_1\ldots b_m\, c_1\ldots c_m}\}$ given by

$$\begin{aligned} f^{(1\ldots m)\mp}_{\{a\}\{b\}\{c\}} &= 2^{-m}\left[\prod_{\gamma=1}^{m}(f^{(\gamma)+}_{a_\gamma b_\gamma c_\gamma} + f^{(\gamma)-}_{a_\gamma b_\gamma c_\gamma}) \mp \prod_{\gamma=1}^{m}(f^{(\gamma)+}_{a_\gamma b_\gamma c_\gamma} - f^{(\gamma)-}_{a_\gamma b_\gamma c_\gamma})\right] \\ &= 2^{-m}\left[\sum_{\sigma_1\cdots\sigma_m=\pm}\left(1 \mp \prod_\gamma \sigma_\gamma\right)\prod_\gamma f^{(\gamma)\sigma_\gamma}_{a_\gamma b_\gamma c_\gamma}\right]\end{aligned} \tag{39}$$

where we have used $f^{\mp}_{bac} = \mp f^{\mp}_{abc}$. For example, for $m = 2$ one finds†

$$f^{(12)_-}_{a_1a_2\,b_1b_2\,c_1c_2} = \tfrac{1}{2}(f^{(1)_+}_{a_1b_1c_1} f^{(2)_-}_{a_2b_2c_2} + f^{(1)_-}_{a_1b_1c_1} f^{(2)_+}_{a_2b_2c_2}). \qquad (40)$$

Each term in $f^{(1\ldots m)_-}$ contains an odd number of $f^{(\gamma)_-}$'s, while each term in $f^{(1\ldots m)_+}$ contains an even number of $f^{(\gamma)_-}$'s, regardless of how many $f^{(\gamma)_+}$'s are involved. Substituting (38) back in (35) we finally obtain

$$\omega \wedge \omega = \frac{1}{2} \sum_{\{a\}}^{\{n\}} \sum_{\{b\}}^{\{n\}} \sum_{\{c\}}^{\{n\}} \bar{\omega}_{\{a\}} \wedge \bar{\omega}_{\{b\}} f^{(1\ldots m)_-}_{\{a\}\{b\}\{c\}} \rho_{\{c\}}. \qquad (41)$$

The whole process, however, can be repeated with respect to any symmetric combination $(\omega_\alpha \wedge \omega_\gamma + \omega_\gamma \wedge \omega_\alpha)$, $\alpha, \gamma = 1, \ldots, m$, all along with the same $f^{(1\ldots m)_-}$'s. Therefore, the passage

$$\sum_{\alpha,\gamma} \rho_E(\omega_\alpha) \wedge \rho_E(\omega_\gamma) \to \sum_{\alpha,\gamma} \rho_E(\omega_\alpha \wedge \omega_\gamma)$$

always involves exactly the same coefficients, as required, and the calculation is completed; the curvature then acquires the form

$$\rho_E(\mathcal{R}_{\mathrm{FC}}(\bar{\omega})) = \sum_{\varpi \in \{\bar{\omega}_\gamma\}} \sum_{\{a\}}^{\{n\}} \sum_{\{b\}}^{\{n\}} \sum_{\{c\}}^{\{n\}} \left(\mathrm{d}\varpi_{\{c\}} + \frac{1}{2} f^{(1\ldots m)_-}_{\{a\}\{b\}\{c\}} \varpi_{\{a\}} \wedge \varpi_{\{b\}} \right) \rho_{\{c\}}. \qquad (42)$$

The foliar complex construction is truly supplied with a group structure which is, however, highly non-trivial because $\otimes_\gamma$ $(\mathrm{Lie}\, G_\gamma) \neq \mathrm{Lie}(\times_\gamma G_\gamma) = \sum_\alpha (\mathrm{Lie}\, G_\alpha)$. This, of course, implies the inequality $f^{(1\ldots m)_-} \neq \prod_\gamma f^{(\gamma)_-}$ and the information about the group structure now lies in the former quantity instead of the latter one. The dimensionality of the grand-group $\mathcal{G} \supset \times_\gamma G_\gamma$ is as large as $\prod_\gamma n_\gamma$ $(n_\gamma \geqslant \dim \mathrm{Lie}\, G_\gamma)$, which is larger than $\dim(\times_\gamma G_\gamma) = \sum_\gamma n_\gamma$. However, the grand-group $\mathcal{G}$ should not be confused with the underlying gauge groups which are the true characteristic symmetries of the splice.

References

[1] Megged O 1996 *J. Phys. A: Math. Gen.* **29** 3233–45

[2] Zumino's approach to ghosts; see Bertlmann R A 1996 *Anomalies in Quantum Field Theory* (Oxford: Oxford Science Publications) pp 347–50, 372–81 and citations therein

† The result is in agreement with that obtained in [1] despite the apparent dissimilarity due to the more crude method that was used there.